Proof of the Completeness Theorem in SL

PHI 201 Introductory Logic
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Here is a description of the proof of the Completeness Theorem that I outlined in class. Much of it follows the text, but since I diverged from the text at some points I thought it might be useful to have it written down.

I will first outline the general strategy that the proof follows, and then fill in the details.

The General Strategy

The theorem we want to prove is this:

Completeness Theorem: If $\Gamma \models \mathbf{P}$ then $\Gamma \vdash \mathbf{P}$.

Now the first important step is to see that in order to prove the Completeness Theorem it suffices to prove the following claim instead:

Principle Lemma: If Γ is consistent in SD then Γ is truth-functionally consistent.[1]

The Principle Lemma implies the Completeness Theorem. To show this, suppose that the Principle Lemma is true and consider a set Γ and a sentence **P** such that $\Gamma \models \mathbf{P}$. We need to show that $\Gamma \vdash \mathbf{P}$. Well, since $\Gamma \models \mathbf{P}$ it follows that $\Gamma \cup \{\sim\mathbf{P}\}$ is not truth-functionally consistent (you proved this in a homework question). But then by the Principle Lemma it follows that $\Gamma \cup \{\sim\mathbf{P}\}$ is not consistent in SD (if you don't see this at first, look at the alternative expression of the Principle Lemma described in footnote 1). But if $\Gamma \cup \{\sim\mathbf{P}\}$ is not consistent in SD then $\Gamma \vdash \mathbf{P}$ (you proved this in a homework question). So $\Gamma \vdash \mathbf{P}$, as required.

[1] In the book, the Principle Lemma is called the 'Inconsistency Lemma' and is numbered 6.4.3. It is expressed there as the claim that if Γ is *not* truth-functionally consistent then Γ is *not* consistent in SD (hence the name they give it). But that is equivalent to what the Principle Lemma states.

So our challenge is to prove the Principle Lemma. To this end, we introduce two notions: that of maximal consistency in SD and that of decomposability.

Definition: A set Γ is *maximally consistent in SD* iff the following two conditions are met:

1. Γ is consistent in SD
2. Given any sentence $\mathbf{P}$ such that $\mathbf{P} \notin \Gamma$, $\Gamma \cup \{\mathbf{P}\}$ is inconsistent in SD.

Definition: A set Γ is *decomposable* iff the following five conditions are met:

1. $\sim\mathbf{P} \in \Gamma$ iff $\mathbf{P} \notin \Gamma$
2. $(\mathbf{P}\ \&\ \mathbf{Q}) \in \Gamma$ iff $\mathbf{P} \in \Gamma$ and $\mathbf{Q} \in \Gamma$
3. $(\mathbf{P} \vee \mathbf{Q}) \in \Gamma$ iff $\mathbf{P} \in \Gamma$ or $\mathbf{Q} \in \Gamma$
4. $(\mathbf{P} \supset \mathbf{Q}) \in \Gamma$ iff either $\mathbf{P} \notin \Gamma$ or $\mathbf{Q} \in \Gamma$
5. $(\mathbf{P} \equiv \mathbf{Q}) \in \Gamma$ iff either both $\mathbf{P} \in \Gamma$ and $\mathbf{Q} \in \Gamma$, or else both $\mathbf{P} \notin \Gamma$ and $\mathbf{Q} \notin \Gamma$

With these two notions in hand, consider the following three claims:

Lemma 1: If a set of sentences Γ is consistent in SD, then there is a set of sentences Δ that is maximally consistent in SD such that $\Gamma \subseteq \Delta$.[2]

Lemma 2: If a set of sentences Γ is maximally consistent in SD, then Γ is decomposable.[3]

Lemma 3: If a set of sentences Γ is decomposable, then Γ is truth-functionally consistent.[4]

Lemmas 1–3 together imply the Principle Lemma. So if we can prove Lemmas 1–3, we're done.

[2] In the book, this is called the 'Maximal Consistency Lemma' and is numbered 6.4.5. In some other texts it is referred to as Lindenbaum's Lemma.

[3] This is what I called the 'Maximal Consistency Theorem' in class.

[4] This is what I called the 'Decomposability Theorem' in class. In the book, Lemmas 2 and 3 are combined into one claim which they call the 'Consistency Lemma' and which is numbered 6.4.8. In some other texts, the Consistency Lemma is referred to as the Truth Lemma.

To see that Lemmas 1–3 together imply the Principle Lemma, consider a set Γ that is consistent in SD. We need to show that if Lemmas 1–3 are all true, then it follows that Γ is truth-functionally consistent. Well, by Lemma 1 it follows that there is a set Δ that is maximally consistent in SD such that $\Gamma \subseteq \Delta$. And since Δ is maximally consistent, it follows from Lemma 2 that Δ is also decomposable. Finally, since Δ is decomposable it follows from Lemma 3 that Δ is also truth-functionally consistent. So there is a truth-value assignment that makes all members of Δ true (by the definition of truth-functional consistency). But $\Gamma \subseteq \Delta$; that is to say, every member of Γ is also a member of Δ. So any truth-value assignment that makes all members of Δ true thereby makes all members of Γ true. Therefore, there is a truth-value assignment that makes all members of Γ true. So Γ is truth-functionally consistent (by the definition of truth-functional consistency). Q.E.D.

To summarize: We want to prove the Completeness Theorem. We first noticed that the Principle Lemma implies the Completeness theorem, so our challenge became that of proving the Principle Lemma. We then noticed that Lemmas 1–3 together imply the Principle Lemma. So in order to prove the Completeness Theorem, all we need to do is prove Lemmas 1–3. Let's take each Lemma in turn.

Lemma 1

Lemma 1: If a set of sentences Γ is consistent in SD, then there is a set of sentences Δ that is maximally consistent in SD such that $\Gamma \subseteq \Delta$.

Proof of Lemma 1: We will show that given any set Γ that is consistent in SD, one can construct a set Δ with the required properties. To construct the set, we start by enumerating the sentences of SL. That is, we assign to each sentence a natural number in such a way that no two sentences are assigned the same number.[5] Given such an enumeration we can inductively define the series of sets Δ_1, Δ_2, Δ_3,... as follows:

i. $\Delta_1 = \Gamma$

ii. $\Delta_{n+1} = \Delta_n \cup \{\mathbf{P}_n\}$ if that is consistent in SD, or Δ_n otherwise.

Finally, we let Δ be the union of all the sets Δ_1, Δ_2, Δ_3,... Our challenge is then to show that Δ has the required properties. That is, we need to show that

[5] One needs to show that it is possible to enumerate the sentences of SL, but I won't outline that here.

1. $\Gamma \subseteq \Delta$.

2. Δ is maximally consistent in SD.

Proof of 1: Easy! Since Δ is defined to be the union of all the sets Δ_1, Δ_2, Δ_3,..., and since $\Delta_1 = \Gamma$, it follows that all members of Γ are also members of Δ, i.e. that $\Gamma \subseteq \Delta$. Q.E.D.

Proof of 2: To show that Δ satisfies the definition of maximal consistency in SD, we need to show that

2.1 Δ is consistent in SD.

2.2 Given any sentence **P** such that $\mathbf{P} \notin \Delta$, $\Delta \cup \{\mathbf{P}\}$ is inconsistent in SD.

Proof of 2.1: First, let us show by induction that each set in the sequence Δ_1, Δ_2, Δ_3,... is consistent in SD. The basis case requires us to show that Δ_1 is consistent in SD, but this is trivial since $\Delta_1 = \Gamma$ and Γ is consistent in SD by hypothesis. The inductive step requires us to show that for any $i \geq 1$, if Δ_i is consistent in SD then Δ_{i+1} is consistent in SD. Well, suppose (as our Inductive Hypothesis) that Δ_i is consistent in SD. We have to show that that Δ_{i+1} is consistent in SD. But this is straightforward, since Δ_{i+1} is defined to be $\Delta_i \cup \{\mathbf{P}_i\}$ if that is consistent in SD, or Δ_i otherwise. In the first case Δ_{i+1} is obviously consistent in SD, and in the second case our Inductive Hypothesis that Δ_i is consistent in SD implies that Δ_{i+1} is consistent in SD. Therefore, each set in the sequence Δ_1, Δ_2, Δ_3,... is consistent in SD.

This does not imply that Δ is consistent in SD, since the union of consistent sets is not always itself consistent. To continue our proof of 2.1, then, let us suppose for reductio that Δ is inconsistent in SD and show that a contradiction follows (in particular, we will show that it follows that one of the sets in the series Δ_1, Δ_2, Δ_3,... is inconsistent in SD, which we have just shown to be false). To this end, note that if Δ is inconsistent in SD then there is a derivation of the following form

1	$\mathbf{Q}_1$	Assumption
2	$\mathbf{Q}_2$	Assumption
	...	
n	$\mathbf{Q}_n$	Assumption
	...	
n+m	**R**	
n+m+1	$\sim$**R**	

where $\mathbf{Q}_1$ through $\mathbf{Q}_n$ are all members of Δ. (We know that there are only finitely many primary assumptions $\mathbf{Q}_1$ through $\mathbf{Q}_n$ because every derivation in SD is finitely long.) Now, let $\mathbf{Q}_k$ be the member of $\{\mathbf{Q}_1, \mathbf{Q}_2, \ldots, \mathbf{Q}_n\}$ that was assigned the highest number in our original enumeration of the sentences of SL. Given how we constructed the series of sets Δ_1, Δ_2, $\Delta_3, \ldots$, it is clear that $\{\mathbf{Q}_1, \mathbf{Q}_2, \ldots, \mathbf{Q}_n\} \subseteq \Delta_{k+1}$. But then since the above derivation is a derivation of a sentence and its negation from members of Δ_{k+1}, it follows that Δ_{k+1} is inconsistent in SD. This contradicts what we proved in the last paragraph, namely that each member of the series Δ_1, Δ_2, $\Delta_3, \ldots$, which includes Δ_{k+1}, is consistent in SD. So our supposition that Δ is inconsistent in SD leads to contradiction and we can conclude that Δ is consistent in SD, as required. Q.E.D.

Proof of 2.2: What we need to show is that given any sentence $\mathbf{P}$, either $\mathbf{P} \in \Delta$ or else $\Delta \cup \{\mathbf{P}\}$ is inconsistent in SD. To this end, remember that $\mathbf{P}$ will have been assigned some number i in our original enumeration, so $\mathbf{P} = \mathbf{P}_i$. Now, $\Delta_i \cup \{\mathbf{P}_i\}$ is either consistent in SD or inconsistent in SD. In the first case, it follows by the definition of Δ_{i+1} that $\mathbf{P}_i \in \Delta_{i+1}$ and then by the definition of Δ that $\mathbf{P}_i \in \Delta$, as required. In the second case, there exists a derivation of the form

1	$\mathbf{Q}_1$	Assumption
2	$\mathbf{Q}_2$	Assumption
	...	
n	$\mathbf{Q}_n$	Assumption
	...	
n+m	$\mathbf{R}$	
n+m+1	$\sim\mathbf{R}$	

where $\mathbf{Q}_1$ through $\mathbf{Q}_n$ are all members of $\Delta_i \cup \{\mathbf{P}_i\}$. But since $\Delta_i \subseteq \Delta$ by the definition of Δ, it follows that $\Delta_i \cup \{\mathbf{P}_i\} \subseteq \Delta \cup \{\mathbf{P}_i\}$. So the above derivation is a derivation of a sentence and its negation from members of $\Delta \cup \{\mathbf{P}_i\}$; hence $\Delta \cup \{\mathbf{P}_i\}$ is inconsistent in SD, as required. Q.E.D.

Lemma 2

Lemma 2: If a set of sentences Γ is maximally consistent in SD, then Γ is decomposable.

To prove this claim, it helps to have two other sub-lemmas in hand:

Lemma 2.1: If a set of sentences $\Gamma \cup \{\mathbf{P}\}$ is inconsistent in SD, then $\Gamma \vdash \sim\mathbf{P}$.[6]

Lemma 2.2 If a set of sentences Γ is maximally consistent in SD and $\Gamma \vdash \mathbf{P}$, then $\mathbf{P} \in \Gamma$.[7]

Let's start by proving these two sub-lemmas, and then turn to proving Lemma 2.

Proof of Lemma 2.1: Suppose that a set of sentences $\Gamma \cup \{\mathbf{P}\}$ is inconsistent in SD. Then by the definition of inconsistency in SD there must exist a derivation of the following form:

1	$\mathbf{Q}_1$	Assumption
2	$\mathbf{Q}_2$	Assumption
	...	
n	$\mathbf{Q}_n$	Assumption
n+1	$\mathbf{P}$	Assumption
	...	
n+m	$\mathbf{R}$	
n+m+1	$\sim\mathbf{R}$	

where $\mathbf{Q}_1$ through $\mathbf{Q}_n$ are all members of Γ. But then it follows that there is also a derivation of the following form:

[6] In the book this claim is numbered 6.4.10.
[7] Although not exactly the same, this corresponds to 6.4.9 in the book.

1	$\mathbf{Q}_1$	Assumption
2	$\mathbf{Q}_2$	Assumption
	...	
n	$\mathbf{Q}_n$	Assumption
n+1	$\mathbf{P}$	Assumption
	...	
n+m	$\mathbf{R}$	
n+m+1	$\sim\mathbf{R}$	
n+m+2	$\sim\mathbf{P}$	lines n+1 through n+m+1, $\sim$ Introduction

Since $\mathbf{Q}_1$ through $\mathbf{Q}_n$ are all members of Γ, it follows that $\Gamma \vdash \sim\mathbf{P}$ by the definition of $\vdash$. Q.E.D.

Proof of Lemma 2.2 Consider a set of sentences Γ that is maximally consistent in SD, and suppose that $\Gamma \vdash \mathbf{P}$. We must show that $\mathbf{P} \in \Gamma$. Well, suppose for reductio that $\mathbf{P} \notin \Gamma$. Then since Γ is maximally consistent in SD, it follows that $\Gamma \cup \{\mathbf{P}\}$ is inconsistent in SD. But then it follows by Lemma 2.1, which we just proved, that $\Gamma \vdash \sim\mathbf{P}$. Since $\Gamma \vdash \mathbf{P}$ by hypothesis, it follows that Γ is inconsistent in SD (by definition of consistency in SD), which contradicts the fact that Γ is maximally consistent. Our supposition that $\mathbf{P} \notin \Gamma$ therefore leads to a contradiction, so we can conclude that $\mathbf{P} \in \Gamma$.

Now that we have these two sub-lemmas in hand, we can turn to proving Lemma 2 itself.

Proof of Lemma 2: Let Γ be a set of sentences that is maximally consistent in SD. We must show that Γ is decomposable. By the definition of decomposability, we must show that it satisfies the following 5 conditions:

1. $\sim\mathbf{P} \in \Gamma$ iff $\mathbf{P} \notin \Gamma$
2. $(\mathbf{P}\ \&\ \mathbf{Q}) \in \Gamma$ iff $\mathbf{P} \in \Gamma$ and $\mathbf{Q} \in \Gamma$
3. $(\mathbf{P} \vee \mathbf{Q}) \in \Gamma$ iff $\mathbf{P} \in \Gamma$ or $\mathbf{Q} \in \Gamma$
4. $(\mathbf{P} \supset \mathbf{Q}) \in \Gamma$ iff either $\mathbf{P} \notin \Gamma$ or $\mathbf{Q} \in \Gamma$
5. $(\mathbf{P} \equiv \mathbf{Q}) \in \Gamma$ iff either both $\mathbf{P} \in \Gamma$ and $\mathbf{Q} \in \Gamma$, or else both $\mathbf{P} \notin \Gamma$ and $\mathbf{Q} \notin \Gamma$

Here I will prove that Γ satisfies conditions 1, 2 and 4. Like the book, I will leave the proof that Γ satisfies conditions 3 and 5 for a homework exercise.

Condition 1: We must show that $\sim\mathbf{P} \in \Gamma$ iff $\mathbf{P} \notin \Gamma$.

In the left-to-right direction, let us suppose that $\sim\mathbf{P} \in \Gamma$. We must show that $\mathbf{P} \notin \Gamma$. To this end let us suppose for reductio that $\mathbf{P} \in \Gamma$ and show that a contradiction follows. Well, consider the following derivation:

1	**P**	Assumption
2	**P**	1, Reiteration

Since we are supposing that $\mathbf{P} \in \Gamma$, this is a derivation of **P** from assumptions that are members of Γ. Therefore $\Gamma \vdash \mathbf{P}$ by the definition of $\vdash$. But since it is also the case that $\sim\mathbf{P} \in \Gamma$, it follows by similar reasoning that $\Gamma \vdash \sim\mathbf{P}$ too. But if $\Gamma \vdash \mathbf{P}$ and $\Gamma \vdash \sim\mathbf{P}$ then Γ is inconsistent in SD (by the definition of inconsistency in SD), which contradicts the fact that Γ is maximally consistent. Our supposition that $\mathbf{P} \in \Gamma$ therefore leads to a contradiction and we can conclude that $\mathbf{P} \notin \Gamma$, as required.

In the right-to-left direction, let us suppose that $\mathbf{P} \notin \Gamma$. We must show that $\sim\mathbf{P} \in \Gamma$. Well, if $\mathbf{P} \notin \Gamma$ then since Γ is maximally consistent in SD it follows by the definition of maximal consistency that $\Gamma \cup \{\mathbf{P}\}$ is inconsistent in SD. So it follows by Lemma 2.1 that $\Gamma \vdash \sim\mathbf{P}$, and then by Lemma 2.2 that $\sim\mathbf{P} \in \Gamma$, as required. Q.E.D.

Condition 2: We must show that $(\mathbf{P} \,\&\, \mathbf{Q}) \in \Gamma$ iff $\mathbf{P} \in \Gamma$ and $\mathbf{Q} \in \Gamma$.

In the left-to-right direction, let us suppose that $(\mathbf{P} \,\&\, \mathbf{Q}) \in \Gamma$. We must then show that $\mathbf{P} \in \Gamma$ and $\mathbf{Q} \in \Gamma$. Well, consider the following derivation:

1	(**P** & **Q**)	Assumption
2	**P**	1, &–Elimination
3	**Q**	1, &–Elimination

Since $(\mathbf{P} \,\&\, \mathbf{Q}) \in \Gamma$ it follows by definition of $\vdash$ that $\Gamma \vdash \mathbf{P}$ and that $\Gamma \vdash \mathbf{Q}$. And since Γ is maximally consistent in SD, it then follows from Lemma 2.2 that $\mathbf{P} \in \Gamma$ and $\mathbf{Q} \in \Gamma$, as required.

In the right-to-left direction, let us suppose that $\mathbf{P} \in \Gamma$ and $\mathbf{Q} \in \Gamma$. We must then show that $(\mathbf{P} \,\&\, \mathbf{Q}) \in \Gamma$. Well, consider the following derivation:

1	**P**	Assumption
2	**Q**	Assumption
3	(**P** & **Q**)	1, 2, &–Introduction

Since **P** $\in \Gamma$ and **Q** $\in \Gamma$, it follows by definition of $\vdash$ that $\Gamma \vdash$ (**P** & **Q**). And since Γ is maximally consistent, it then follows from Lemma 2.2 that (**P** & **Q**) $\in \Gamma$, as required. Q.E.D.

Condition 4: We must show that (**P** $\supset$ **Q**) $\in \Gamma$ iff either **P** $\notin \Gamma$ or **Q** $\in \Gamma$.

In the left-to-right direction, let us suppose that (**P** $\supset$ **Q**) $\in \Gamma$. We must then show that **P** $\notin \Gamma$ or **Q** $\in \Gamma$. Well, either **P** $\in \Gamma$ or **P** $\notin \Gamma$. If **P** $\notin \Gamma$ then we're done, so the interesting case is the case in which **P** $\in \Gamma$; we will show that it follows from this that **Q** $\in \Gamma$. Well, consider the following derivation:

1	(**P** $\supset$ **Q**)	Assumption
2	**P**	Assumption
3	**Q**	1, 2, $\supset$–Elimination

Since (**P** $\supset$ **Q**) $\in \Gamma$, and since we are considering the case in which **P** $\in \Gamma$, both assumptions are members of Γ and therefore $\Gamma \vdash$ **Q** by the definition of $\vdash$. And since Γ is maximally consistent in SD, it follows by Lemma 2.2 that **Q** $\in \Gamma$, as required.

In the left-to-right direction, let us suppose that **P** $\notin \Gamma$ or **Q** $\in \Gamma$. We need to show that in either case, (**P** $\supset$ **Q**) $\in \Gamma$. Well, start with the case in which **P** $\notin \Gamma$. Since Γ is maximally consistent in SD, $\Gamma \cup \{$**P**$\}$ is inconsistent in SD (by the definition of maximal consistency). So $\Gamma \vdash$ $\sim$**P** (by Lemma 2.1), and hence $\sim$**P** $\in \Gamma$ (by Lemma 2.2). But now consider the following derivation:

1	$\sim\mathbf{P}$	Assumption
2	$\mathbf{P}$	Assumption
3	$\sim\mathbf{Q}$	Assumption
4	$\mathbf{P}$	2, Reiteration
5	$\sim\mathbf{P}$	1, Reiteration
6	$\mathbf{Q}$	3–5, $\sim$–Elimination
7	$(\mathbf{P} \supset \mathbf{Q})$	2–6, $\supset$–Introduction

Since $\sim\mathbf{P} \in \Gamma$, it follows from the definition of $\vdash$ that $\Gamma \vdash (\mathbf{P} \supset \mathbf{Q})$, and then from Lemma 2.2 that $(\mathbf{P} \supset \mathbf{Q}) \in \Gamma$, as required. It remains to show that in the case that $\mathbf{Q} \in \Gamma$, it still follows that $(\mathbf{P} \supset \mathbf{Q})$. Well, consider the following derivation:

1	$\mathbf{Q}$	Assumption
2	$\mathbf{P}$	Assumption
3	$\mathbf{Q}$	1, Reiteration
4	$(\mathbf{P} \supset \mathbf{Q})$	2–3, $\supset$–Introduction

Since we are considering the case in which $\mathbf{Q} \in \Gamma$, it follows from the definition of $\vdash$ that $\Gamma \vdash (\mathbf{P} \supset \mathbf{Q})$ and then from Lemma 2.2 that $(\mathbf{P} \supset \mathbf{Q}) \in \Gamma$, as required. Q.E.D.

Lemma 3

Lemma 3: If a set of sentences Γ is decomposable, then Γ is truth-functionally consistent.

To prove this we will show that given a decomposable set Γ, one can construct a truth-value assignment that makes all the members of Γ true. Here is how the truth-value assignment is constructed:

Definition: Given a set Γ of sentences of SL let a_Γ be the truth value assignment defined as follows: for all sentence letters **P**, a_Γ assigns truth to **P** iff $\mathbf{P} \in \Gamma$.

We will prove that given any decomposable set Γ, the truth-value assignment a_Γ makes all members of Γ true. Actually, we will prove something a little more general:

Lemma 3.1: If a set Γ of sentences is decomposable, then a_Γ makes **P** true iff $\mathbf{P} \in \Gamma$.

(That is, a_Γ makes true all the sentences in Γ and does not make true any other sentences.)

Clearly, Lemma 3.1 implies Lemma 3. For suppose that Γ is decomposable. Then by Lemma 3.1 it follows that a_Γ makes **P** true iff $\mathbf{P} \in \Gamma$. That is, a_Γ makes true all the sentences in Γ, so by the definition of truth-functional consistency Γ is truth-functionally consistent as Lemma 3 states. So if we prove Lemma 3.1, we will have thereby proved Lemma 3.

Proof of Lemma 3.1: By induction on sentence complexity. We prove that for any decomposable set Γ the following two claims are true:

- Basis Case: If **P** is a sentence letter, then a_Γ makes **P** true iff $\mathbf{P} \in \Gamma$.
- Inductive Step: If **P** and **Q** are both sentences such that (i) a_Γ makes **P** true iff $\mathbf{P} \in \Gamma$, and (ii) a_Γ makes **Q** true iff $\mathbf{Q} \in \Gamma$, then
  1. a_Γ makes $\sim\mathbf{P}$ true iff $\sim\mathbf{P} \in \Gamma$,
  2. a_Γ makes $(\mathbf{P} \,\&\, \mathbf{Q})$ true iff $(\mathbf{P} \,\&\, \mathbf{Q}) \in \Gamma$,
  3. a_Γ makes $(\mathbf{P} \vee \mathbf{Q})$ true iff $(\mathbf{P} \vee \mathbf{Q}) \in \Gamma$,
  4. a_Γ makes $(\mathbf{P} \supset \mathbf{Q})$ true iff $(\mathbf{P} \supset \mathbf{Q}) \in \Gamma$,
  5. a_Γ makes $(\mathbf{P} \equiv \mathbf{Q})$ true iff $(\mathbf{P} \equiv \mathbf{Q}) \in \Gamma$,

Proof of Basis Case: Easy! Let **P** be a sentence letter. Then a_Γ makes **P** true iff $\mathbf{P} \in \Gamma$ by the definition of a_Γ.

Proof of Inductive Step: Our Inductive Hypothesis (IH) is that **P** and **Q** are indeed both sentences such that (i) a_Γ makes **P** true iff $\mathbf{P} \in \Gamma$, and (ii) a_Γ makes **Q** true iff $\mathbf{Q} \in \Gamma$. We then need to prove that 1–5 obtain.

Proof of 1: If a_Γ makes $\sim\mathbf{P}$ true then a_Γ makes **P** false (by truth table for $\sim$). So $\mathbf{P} \notin \Gamma$ (by IH). So $\sim\mathbf{P} \in \Gamma$ (by the fact that Γ is decomposable). Same reasoning works in reverse.

Proof of 2: If a_Γ makes $(\mathbf{P} \,\&\, \mathbf{Q})$ true then a_Γ makes **P** true and a_Γ makes **Q** true (by the truth table for &). So $\mathbf{P} \in \Gamma$ and $\mathbf{Q} \in \Gamma$ (by IH). So $(\mathbf{P} \,\&\, \mathbf{Q}) \in \Gamma$ (by the fact that Γ is decomposable). Same reasoning works in reverse.

Proof of 3: If a_Γ makes ($\mathbf{P} \vee \mathbf{Q}$) true then a_Γ makes **P** true or a_Γ makes **Q** true (by the truth table for $\vee$). So $\mathbf{P} \in \Gamma$ or $\mathbf{Q} \in \Gamma$ (by IH). So ($\mathbf{P} \vee \mathbf{Q}$) $\in \Gamma$ (by the fact that Γ is decomposable). Same reasoning works in reverse.

Proof of 4: If a_Γ makes ($\mathbf{P} \supset \mathbf{Q}$) true then either a_Γ makes **P** false or a_Γ makes **Q** true (by the truth table for $\supset$). So either $\mathbf{P} \notin \Gamma$ or $\mathbf{Q} \in \Gamma$ (by IH). So ($\mathbf{P} \supset \mathbf{Q}$) $\in \Gamma$ (by the fact that Γ is decomposable). Same reasoning works in reverse.

Proof of 5: If a_Γ makes ($\mathbf{P} \equiv \mathbf{Q}$) true then either a_Γ makes both **P** and **Q** true, or else a_Γ makes both **P** and **Q** false (by the truth table for $\equiv$). So either both $\mathbf{P} \in \Gamma$ and $\mathbf{Q} \in \Gamma$, or else $\mathbf{P} \notin \Gamma$ and $\mathbf{Q} \notin \Gamma$ (by IH). So ($\mathbf{P} \equiv \mathbf{Q}$) $\in \Gamma$ (by the fact that Γ is decomposable). Same reasoning works in reverse. Q.E.D.