Sentential Logic

PHI 201 Introductory Logic

Fall 2012

This is a summary of definitions in Sentential Logic (SL) from the text *The Logic Book* by Bergmann *et al*.

1 The Language SL

Vocabulary

The *vocabulary of SL* consists in the following:

1. Sentence letters: A, B,. . . , Z, A_1, B_1,. . . , Z_1, A_2, B_2,. . .
2. Connectives: $\sim$, &, $\vee$, $\supset$, $\equiv$
3. Punctuation:), (

Sentences

The *sentences of SL* are defined as follows:

1. Every sentence letter of SL is a sentence of SL.
2. If **P** is a sentence of SL then $\sim$**P** is a sentence of SL.
3. If **P** and **Q** are sentences of SL then (**P** & **Q**) is a sentence of SL.
4. If **P** and **Q** are sentences of SL then (**P** $\vee$ **Q**) is a sentence of SL.
5. If **P** and **Q** are sentences of SL then (**P** $\supset$ **Q**) is a sentence of SL.
6. If **P** and **Q** are sentences of SL then (**P** $\equiv$ **Q**) is a sentence of SL.
7. That's all, folks! Nothing else is a sentence of SL.

2 Semantics

Truth

A *truth-value assignment* is an assignment of a truth-value, either the value **T** (true) or the value **F** (false) but not both, to each sentence letter of SL.

Given a truth-value assignment, the truth-values of all sentences of SL are determined by the following tables:

P	**Q**	**(P & Q)**
T	**T**	**T**
T	**F**	**F**
F	**T**	**F**
F	**F**	**F**

P	**Q**	**(P ∨ Q)**
T	**T**	**T**
T	**F**	**T**
F	**T**	**T**
F	**F**	**F**

P	**∼P**
T	**F**
F	**T**

P	**Q**	**(P ⊃ Q)**
T	**T**	**T**
T	**F**	**F**
F	**T**	**T**
F	**F**	**T**

P	**Q**	**(P ≡ Q)**
T	**T**	**T**
T	**F**	**F**
F	**T**	**F**
F	**F**	**T**

Truth-Functional Concepts

A sentence **P** of SL is *truth-functionally true* iff **P** is true on every truth-value assignment.

A sentence **P** of SL is *truth-functionally false* iff **P** is false on every truth-value assignment.

A sentence **P** of SL is *truth-functionally indeterminate* iff **P** is neither truth-functionally true nor truth-functionally false.

Sentences **P** and **Q** of SL are *truth-functionally equivalent* iff there is no truth-value assignment on which **P** and **Q** have different truth-values.

A set Γ of sentences of SL is *truth-functionally consistent* iff there is a truth-value assignment on which all members of Γ are true.

A set Γ of sentences of SL *truth-functionally entails* a sentence **P** of SL, written $\Gamma \models \mathbf{P}$, iff there is no truth-value assignment on which all members of Γ are true and **P** is false.

An argument of SL is *truth-functionally valid* iff there is no truth-value assignment on which all the premises are true and the conclusion is false.

3 Proof

The Rules of SD

The following rules constitute the derivation system SD:

R

$$\begin{array}{r|l} & \mathbf{P} \\ \Longrightarrow & \mathbf{P} \end{array}$$

& Intro

$$\begin{array}{r|l} & \mathbf{P} \\ & \mathbf{Q} \\ \Longrightarrow & (\mathbf{P} \,\&\, \mathbf{Q}) \end{array}$$

& Exit

$$\begin{array}{r|l} & (\mathbf{P} \,\&\, \mathbf{Q}) \\ \Longrightarrow & \mathbf{P} \end{array}$$

or

$$\begin{array}{r|l} & (\mathbf{P} \,\&\, \mathbf{Q}) \\ \Longrightarrow & \mathbf{Q} \end{array}$$

⊃ Intro

$$\begin{array}{r|l} & \begin{array}{|l} \mathbf{P} \\ \hline \\ \mathbf{Q} \end{array} \\ \Longrightarrow & (\mathbf{P} \supset \mathbf{Q}) \end{array}$$

⊃ Exit

$$\begin{array}{r|l} & (\mathbf{P} \supset \mathbf{Q}) \\ & \mathbf{P} \\ \Longrightarrow & \mathbf{Q} \end{array}$$

∼ Intro

$$\begin{array}{r|l} & \begin{array}{|l} \mathbf{P} \\ \hline \\ \mathbf{Q} \\ \sim\mathbf{Q} \end{array} \\ \Longrightarrow & \sim\mathbf{P} \end{array}$$

∼ Exit

$$\begin{array}{r|l} & \begin{array}{|l} \sim\mathbf{P} \\ \hline \\ \mathbf{Q} \\ \sim\mathbf{Q} \end{array} \\ \Longrightarrow & \mathbf{P} \end{array}$$

$\vee$ **Intro**

$$\begin{array}{r|l} & \mathbf{P} \\ \Longrightarrow & (\mathbf{P} \vee \mathbf{Q}) \end{array} \qquad \text{or} \qquad \begin{array}{r|l} & \mathbf{P} \\ \Longrightarrow & (\mathbf{Q} \vee \mathbf{P}) \end{array}$$

$\vee$ **Exit**

$$\begin{array}{r|l} & (\mathbf{P} \vee \mathbf{Q}) \\ & \begin{array}{|l} \mathbf{P} \\ \hline \\ \mathbf{R} \end{array} \\ & \begin{array}{|l} \mathbf{Q} \\ \hline \\ \mathbf{R} \end{array} \\ \Longrightarrow & \mathbf{R} \end{array}$$

$\equiv$ **Intro**

$$\begin{array}{r|l} & \begin{array}{|l} \mathbf{P} \\ \hline \\ \mathbf{Q} \end{array} \\ & \begin{array}{|l} \mathbf{Q} \\ \hline \\ \mathbf{P} \end{array} \\ \Longrightarrow & (\mathbf{P} \equiv \mathbf{Q}) \end{array}$$

$\equiv$ **Exit**

$$\begin{array}{r|l} & (\mathbf{P} \equiv \mathbf{Q}) \\ & \mathbf{P} \\ \Longrightarrow & \mathbf{Q} \end{array} \qquad \text{or} \qquad \begin{array}{r|l} & (\mathbf{P} \equiv \mathbf{Q}) \\ & \mathbf{Q} \\ \Longrightarrow & \mathbf{P} \end{array}$$

Proof-Theoretic Concepts

A *derivation in SD* is a series of sentences of SL, each of which is either an assumption or is obtained from previous sentences by one of the rules of SD.

A sentence **P** of SL is *derivable in SD* from a set Γ of sentences of SL, written **S** $\vdash$ **P**, iff there exists a derivation in SD in which all the primary assumptions are members of Γ and **P** occurs within the scope of only the primary assumptions.

An argument of SL is *valid in SD* iff the conclusion of the argument is derivable in SD from the set consisting of the premises.

A sentence **P** of SL is a *theorem in SD* iff **P** is derivable in SD from the empty set.

Sentences **P** and **Q** of SL are *equivalent in SD* iff **Q** is derivable from {**P**} and **P** is derivable from {**Q**}.

A set Γ of sentences of SL is *inconsistent in SD* iff there is a sentence **P** such that both **P** and ∼**P** are derivable in SD from Γ.