Predicate Logic

PHI 201 Introductory Logic
Spring 2011

This is a summary of definitions in Predicate Logic from the text *The Logic Book* by Bergmann *et al*.

1 The Language PLE

Vocabulary

The *vocabulary of PLE* consists in the following:

1. Sentence letters: A, B,..., Z, A_1, B_1,..., Z_1, A_2, B_2,...
2. n-place predicates, for any positive integer n: A′, B′,..., Z′, $A_1{}'$, $B_1{}'$,..., $Z_1{}'$... and so on; also A″, B″,..., Z″, $A_1{}''$, $B_1{}''$..., $Z_1{}''$... and so on.
   * In practice, the primes are rarely used and predicates are usually written Ax, Bx,..., Axy, Bxy,... instead.
3. The two-place identity predicate, =.
4. Simple terms:
   (a) Individual constants: a, b,..., v, a_1, b_1..., v_1..., a_2, b_2...
   (b) Individual variables: w, x, y, z, w_1, x_1, y_1, z_1, w_2,...
5. n-place function expressions, for any positive integer n: a', b',..., $a_1{}'$, $b_1{}'$,... and so on; also a'', b'',..., $a_1{}''$, $b_1{}''$,... and so on.
   * Like in the case of predicates, in practice the primes are rarely used and function expressions are usually written a(x), b(x),..., a(x, y), b(x, y),... instead.
6. Connectives: $\sim$, &, $\vee$, $\supset$, $\equiv$
7. Quantifier Symbols: $\forall$, $\exists$
8. Punctuation:), (

Formulas of PLE

A *quantifier of PLE* is an expression of PLE of the form ($\forall$**x**) or ($\exists$**x**).

The *individual terms of PLE* are defined as follows:

1. Every simple term is an individual term of PLE.
2. If $\mathbf{t}_1$, $\mathbf{t}_2$,... $\mathbf{t}_n$ are individual terms of PLE and ***f*** is an n-place function expression, then ***f***($\mathbf{t}_1$, $\mathbf{t}_2$,..., $\mathbf{t}_n$) is an individual term of PLE.

An individual term of PLE is *closed* iff it contains no variables; and is *open* otherwise.

An *atomic formula of PLE* is an expression of PLE that is either a sentence letter of PLE or an n-place predicate of PLE followed by n individual terms of PLE.

The *formulas of PLE* are defined as follows:

1. Every atomic formula of PLE is a formula of PLE.
2. If **P** and **Q** are formulas of PLE then so are $\sim$**P**, (**P** & **Q**), (**P** $\vee$ **Q**), (**P** $\supset$ **Q**), and (**P** $\equiv$ **Q**).
3. If **P** is a formula of PLE that contains at least one occurrence of **x** and no **x**-quantifier then ($\forall$**x**)**P** and ($\exists$**x**)**P** are both formulas of PLE.
4. That's all folks! Nothing else is a formula of PLE.

Sentences of PLE

The *scope of a quantifier* in a formula **P** is the subformula **Q** of **P** of which that quantifier is the main logical operator.

An occurrence of a variable **x** in a formula of PLE is *bound* if it is in the scope of an **x**-quantifier. Otherwise it is *free*.

A formula **P** of PLE is a *sentence of PLE* if no occurrence of a variable in **P** is free.

2 Semantics

Truth

Warning: This is the *informal* semantics presented in Bergmann *et al.* Some important details dealt with by the formal semantics are left implicit.

An *interpretation* is

1. A non-empty universe of discourse (UD),
2. An assignment of an object in the UD to each individual constant, and
3. An assignment of an n-place property to each n-place predicate except the two-place identity predicate =.
4. An assignment of a function, which maps each member of the UD to a member of the UD, to each function expression.

Sentences of PL are defined to be true or false in a given interpretation I as follows:

1. If **F** is an n-place predicate other than = and $\mathbf{t}_1$, $\mathbf{t}_2$,. . . , $\mathbf{t}_n$ are n closed individual terms, the atomic sentence $\mathbf{Ft}_1\mathbf{t}_2\ldots\mathbf{t}_n$ is true in I iff the objects assigned to each closed individual term by I, in that order, have the property assigned to the predicate **F** by I.
2. A sentence of the form $\mathbf{t}_1 = \mathbf{t}_2$ is true in I iff the object assigned to $\mathbf{t}_1$ by I is identical to the object assigned to $\mathbf{t}_2$ by I.
3. A sentence of PL of the form $(\forall \mathbf{x})\mathbf{P}$ is true in I if every member of the UD of I has the property assigned to **P** by I.
4. A sentence of PL of the form $(\exists \mathbf{x})\mathbf{P}$ is true in I if at least one member of the UD of I has the property assigned to **P** by I.
5. Sentences of PL of the form $\sim\mathbf{P}$, $(\mathbf{P}\ \&\ \mathbf{Q})$, $(\mathbf{P} \vee \mathbf{Q})$, $(\mathbf{P} \supset \mathbf{Q})$, and $(\mathbf{P} \equiv \mathbf{Q})$ have their truth values in I determined by the truth-tables for Sentential Logic.

Quantificational Concepts

A sentence **P** of PLE is *quantificationally true* iff **P** is true on every interpretation.

A sentence **P** of PLE is *quantificationally false* iff **P** is false on every interpretation.

A sentence **P** of PLE is *quantificationally indeterminate* iff **P** is neither quantificationally true nor quantificationally false.

Sentences **P** and **Q** of PLE are *quantificationally equivalent* if there is no interpretation on which **P** and **Q** have different truth values.

A set Γ of sentences of PLE is *quantificationally consistent* if there is an interpretation on which every member of Γ is true.

A set Γ of sentences of PLE *quantificationally entails* a sentence **P** of PL, written $\mathbf{S} \models \mathbf{P}$, if there is no interpretation on which every member of Γ is true and **P** is false.

* The sign '$\models$' is also used in SL to stand for truth-functional entailment. When using the sign we will try to make it clear which sense is intended; if it is unclear, please ask.

An argument of PLE is *quantificationally valid* if the premises quantificationally entail the conclusion.

3 Proof

The Rules of PDE

If **P** is a formula of PLE and **t** is a closed individual term of PLE, then **P**(**t**/**x**) is just like **P** except that it contains **t** wherever **P** contains the variable **x**.

The rules of PDE are all those of SD plus the following:

∃-Intro

$$\begin{array}{r|l} & \mathbf{P(t/x)} \\ \Longrightarrow & (\exists \mathbf{x})\mathbf{P} \end{array}$$

∀-Exit

$$\begin{array}{r|l} & (\forall \mathbf{x})\mathbf{P} \\ \Longrightarrow & \mathbf{P(t/x)} \end{array}$$

where **t** is any closed individual term.

∀-Intro

$$\begin{array}{r|l} & \mathbf{P(a/x)} \\ \Longrightarrow & (\forall \mathbf{x})\mathbf{P} \end{array}$$

where **a** is an individual constant, so long as (i) **a** doesn't occur in an undischarged assumptions, and (ii) **a** doesn't occur in (∀**x**)**P**.

∃-Exit

$$
\begin{array}{r|l}
 & (\exists \mathbf{x})\mathbf{P} \\
 & \quad \begin{array}{|l}
 \mathbf{P}(\mathbf{a}/\mathbf{x}) \\ \hline
 \\
 \mathbf{Q}
 \end{array} \\
\Longrightarrow & \mathbf{Q}
\end{array}
$$

where **a** is an individual constant, so long as (i) **a** doesn't occur in an undischarged assumptions, (ii) **a** doesn't occur in $(\exists \mathbf{x})\mathbf{P}$, and (iii) **a** doesn't occur in **Q**.

=-Intro $\Longrightarrow \mid (\forall \mathbf{x})\mathbf{x} = \mathbf{x}$

=-Exit

$$
\begin{array}{r|l}
 & \mathbf{t}_1 = \mathbf{t}_2 \\
 & \mathbf{P} \\
\Longrightarrow & \mathbf{P}(\mathbf{t}_1 // \mathbf{t}_2)
\end{array}
\qquad \text{or} \qquad
\begin{array}{r|l}
 & \mathbf{t}_1 = \mathbf{t}_2 \\
 & \mathbf{P} \\
\Longrightarrow & \mathbf{P}(\mathbf{t}_2 // \mathbf{t}_1)
\end{array}
$$

where $\mathbf{t}_1$ and $\mathbf{t}_2$ are closed individual terms, and $\mathbf{P}(\mathbf{t}_i \;//\; \mathbf{t}_j)$ is **P** with one or more occurrences of $\mathbf{t}_i$ replaced by $\mathbf{t}_j$.

Proof-Theoretic Concepts

A *derivation in PDE* is a series of sentences of PLE, each of which is either an assumption or is obtained from previous sentences by one of the rules of PDE.

A sentence **P** of PLE is *derivable in PDE* from a set Γ of sentences of PLE, written **S** ⊢ **P**, iff there exists a derivation in PDE in which all the primary assumptions are members of Γ and **P** occurs within the scope of only the primary assumptions.

* The sign '⊢' is also used in SD to stand for the notion of being derivable in SD. When using the sign we will try to make it clear which sense is intended; if it is unclear, please ask.

An argument of PLE is *valid in PDE* iff the conclusion of the argument is derivable in PDE from the set consisting of the premises.

A sentence **P** of PLE is a *theorem in PDE* iff **P** is derivable in PDE from the empty set.

Sentences **P** and **Q** of PLE are *equivalent in PDE* iff **Q** is derivable in PDE from {**P**} and **P** is derivable in PDE from {**Q**}.

A set Γ of sentences of PLE is *inconsistent in PDE* iff there is a sentence **P** such that both **P** and ∼**P** are derivable in PDE from Γ.